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FREE TRANSVERSE VIBRATIONS OF ELASTICALLY CONNECTED SIMPLY SUPPORTED DOUBLE-BEAM COMPLEX SYSTEM

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In this paper, the free vibration analysis of two parallel simply supported beams continuously joined by a Winkler elastic layer is presented. The motion of the system is described by a homogeneous set of two partial differential equations, which is solved by using the classical Bernoulli–Fourier method. The natural frequencies of the system are determined. The initial-value problem is considered to find the final form of the free vibrations. The free vibrations of an elastically connected double-beam complex system are realized by synchronous and asynchronous deflections. The presented theoretical analysis is illustrated by a numerical example, in which the effect of physical parameters characterizing the vibrating system on the natural frequencies is investigated.

1. INTRODUCTION

An elastically connected double-beam system is another model of a complex continuous system consisting of two one-dimensional solids joined by linear, elastic layer of a Winkler type. Analogously built double-string system is the simplest model of a complex continuous system. The transverse vibration analysis of this interesting vibratory system has been presented by the author in a few of his early works [1–9]. In the present paper the free vibrations of a double-beam system are considered for the simple case of simply supported boundary conditions. Assumed particular boundary conditions make it possible to solve the vibration problem for the beams by using the less general mathematical procedures. Because of the same boundary conditions the vibration problems for both of these complex continuous systems discussed (a double-string and double-beam system) are similar. Then the vibration analysis of a double-string system can be helpful in the investigation of a system considered.

The different aspects of dynamics of an elastically connected double-beam system have been treated by many authors: Seelig and Hoppmann II [10, 11], Kessel [12], Kessel and Raske [13], Saito and Chonan [14, 15], Kozlov [16], Kashin [17], Rao [18], Lu and Douglas [19], Douglas and Yang [20], Douglas [21], Oniszczuk [22–31], Chonan [32, 33], Hyer *et al.* [34, 35], Stepanov [36], Dmitriyev [37], Hamada *et al.* [38, 39], Yamaguchi and Saito [40], Joshi and Upadhya [41], Sylwan [42], Vaswani *et al.* [43], Kokhmaniuk [44], Yankelevsky [45], Aida *et al.* [46], Frostig and Baruch [47, 48], Kukla and Skalmierski [49], Kukla [50, 51], Macé [52], Chen and Sheu [53, 54], Chen and Lin [55], Lueschen and Bergman [56], Sakiyama *et al.* [57, 58], Szcześniak [59, 60], Kawazoe *et al.* [61], and Cabańska-Płaczkiewicz [62–67]. The works [2, 4, 5, 36, 38–40, 46, 55, 61]

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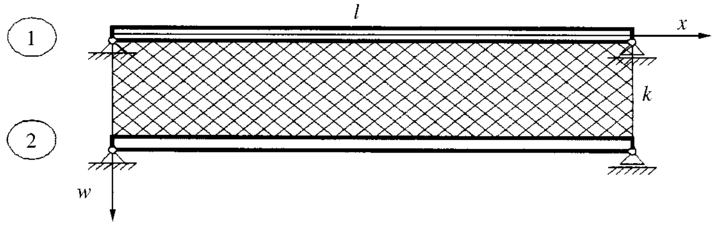

Figure 1. The physical model of an elastically connected double-beam complex system.

devoted to applying a double-beam system as a continuous dynamic vibration absorber (CDVA) are especially interesting because of the great practical importance of CDVAs in many fields of civil and mechanical engineering.

In the present paper, the complete exact theoretical solutions of the free vibrations of simply supported double-beam system are formulated. The general free vibration analysis of an elastically connected double-beam system is complicated and laborious in view of the large variety of possible combinations of the boundary conditions. Therefore, in this work the considerations are limited only to the case of simply supported beams. The vibrations of a general system with the homogeneous boundary conditions will be discussed by the author in the next publications concerning this subject.

2. FORMULATION OF THE PROBLEM

The physical model of the vibrating system under consideration is composed of two parallel, slender, prismatic and homogeneous beams joined by a Winkler elastic layer (see Figure 1). Both beams have the same length and are simply supported at their ends. The small undamped vibrations of the system are considered.

Using the Bernoulli–Euler beam theory the free transverse vibrations of an elastically connected double-beam system are described by the following differential equations [2, 24]:

$$K_1 w_1^{\mathrm{IV}} + m_1 \ddot{w}_1 + k(w_1 - w_2) = 0, \quad K_2 w_2^{\mathrm{IV}} + m_2 \ddot{w}_2 + k(w_2 - w_1) = 0, \tag{1}$$

where $w_i = w_i(x, t)$ is the transverse beam deflection; x, t are the spatial co-ordinate and the time; E_i is the Young modulus of elasticity; F_i is the cross-sectional area of the beam; J_i is the moment of inertia of the beam cross-section; K_i is the flexural rigidity of the beam; k is the stiffness modulus of a Winkler elastic layer; l is the length of the beam and ρ_i is the mass density,

$$K_i = E_i J_i, \quad m_i = \rho_i F_i, \quad \dot{w}_i = \partial w_i / \partial t, \quad w_i' = \partial w_i / \partial x, \quad i = 1, 2.$$

The initial conditions in general form and boundary conditions for simply supported beams are assumed as follows:

$$w_i(0, t) = w_i''(0, t) = w_i(l, t) = w_i''(l, t) = 0, \tag{2}$$

$$w_i(x, 0) = w_{i0}(x), \quad \dot{w}_i(x, 0) = v_{i0}(x), \quad i = 1, 2. \tag{3}$$

3. SOLUTION OF THE PROBLEM

The homogeneous partial differential equations (1) with the governing boundary conditions (2) can be solved by the Bernoulli–Fourier method assuming the solutions in the form

$$w_i(x,t) = \sum_{n=1}^{\infty} X_n(x) S_{in}(t) = \sum_{n=1}^{\infty} \sin(k_n x) S_{in}(t), \quad i = 1, 2, \tag{4}$$

where $S_{in}(t)$ is the unknown time function;

$$X_n(x) = \sin(k_n x), \quad k_n = l^{-1} n\pi, \quad n = 1,2,3,\ldots. \tag{5}$$

$X_n(x)$ is the known mode shape function for simply supported single beam.

Introducing the general solutions (4) into equations (1) one gets the following relations:

$$\sum_{n=1}^{\infty} [\ddot{S}_{1n} + (K_1 k_n^4 + k) m_1^{-1} S_{1n} - k m_1^{-1} S_{2n}] X_n = 0,$$

$$\sum_{n=1}^{\infty} [\ddot{S}_{2n} + (K_2 k_n^4 + k) m_2^{-1} S_{2n} - k m_2^{-1} S_{1n}] X_n = 0.$$

From the above one obtains a set of ordinary differential equations for the unknown time functions

$$\ddot{S}_{1n} + \Omega_{11n}^2 S_{1n} - \Omega_{10}^2 S_{2n} = 0, \quad \ddot{S}_{2n} + \Omega_{22n}^2 S_{2n} - \Omega_{20}^2 S_{1n} = 0, \tag{6}$$

where

$$\Omega_{iin}^2 = (K_i k_n^4 + k) m_i^{-1}, \quad \Omega_{i0}^2 = k m_i^{-1}, \quad \Omega_{120}^4 = \Omega_{10}^2 \Omega_{20}^2 = k^2 (m_1 m_2)^{-1}, \quad i = 1, 2,$$

Ω_{iin} and Ω_{120} denote partial and coupling frequency of the system respectively. The solutions of equations (6) are as follows:

$$S_{1n}(t) = C_n \mathrm{e}^{\mathrm{i}\omega_n t}, \quad S_{2n}(t) = D_n \mathrm{e}^{\mathrm{i}\omega_n t}, \quad \mathrm{i} = (-1)^{1/2}, \tag{7}$$

where ω_n is the natural frequency of the system. By substituting them into equations (6) results in the following system of homogeneous algebraic equations for the unknown constants C_n, D_n:

$$(\Omega_{11n}^2 - \omega_n^2) C_n - \Omega_{10}^2 D_n = 0, \quad (\Omega_{22n}^2 - \omega_n^2) D_n - \Omega_{20}^2 C_n = 0. \tag{8}$$

Equations (8) have non-trivial solutions in the case when the determinant of the system coefficient matrix is equal to zero. This yields the following frequency (characteristic) equation:

$$\omega_n^4 - (\Omega_{11n}^2 + \Omega_{22n}^2)\omega_n^2 + (\Omega_{11n}^2 \Omega_{22n}^2 - \Omega_{120}^4) = 0 \tag{9}$$

or

$$\omega_n^4 - [(K_1 k_n^4 + k) m_1^{-1} + (K_2 k_n^4 + k) m_2^{-1}] \omega_n^2 + k_n^4 [K_1 K_2 k_n^4 + k(K_1 + K_2)] (m_1 m_2)^{-1} = 0. \tag{10}$$

Since the discriminant of this biquadratic algebraic equation is positive

$$D = (\Omega_{11n}^2 + \Omega_{22n}^2)^2 - 4(\Omega_{11n}^2 \Omega_{22n}^2 - \Omega_{120}^4) = (\Omega_{11n}^2 - \Omega_{22n}^2)^2 + 4\Omega_{120}^4 > 0$$

and the relationships mentioned below are also satisfied

$$(\Omega_{11n}^2\Omega_{22n}^2 - \Omega_{120}^4) > 0, \quad (\Omega_{11n}^2 + \Omega_{22n}^2) > D^{1/2},$$

then the characteristic equation (9) has two different, real, positive roots $\omega_{1,2n}^2$:

$$\omega_{1,2n}^2 = 0{\cdot}5\,\{(\Omega_{11n}^2 + \Omega_{22n}^2) \mp [(\Omega_{11n}^2 - \Omega_{22n}^2)^2 + 4\Omega_{120}^4]^{1/2}\}, \quad \omega_{1n} < \omega_{2n}. \qquad (11)$$

One obtains two infinite sequences of natural frequencies ω_{1n}, ω_{2n} $(\omega_{1n} < \omega_{2n})$ in the form

$$\omega_{1,2n}^2 = 0{\cdot}5\,\{[(K_1k_n^4 + k)m_1^{-1} + (K_2k_n^4 + k)m_2^{-1}] \mp ([(K_1k_n^4 + k)m_1^{-1}$$

$$+ (K_2k_n^4 + k)m_2^{-1}]^2 - 4k_n^4(m_1m_2)^{-1}[K_1K_2k_n^4 + k(K_1 + K_2)])^{1/2}\}. \qquad (12)$$

Now the solutions (7) may be written as

$$S_{1n}(t) = C_{1n}e^{i\omega_{1n}t} + C_{2n}e^{-i\omega_{1n}t} + C_{3n}e^{i\omega_{2n}t} + C_{4n}e^{-i\omega_{2n}t},$$

$$S_{2n}(t) = D_{1n}e^{i\omega_{1n}t} + D_{2n}e^{-i\omega_{1n}t} + D_{3n}e^{i\omega_{2n}t} + D_{4n}e^{-i\omega_{2n}t}.$$

After rearranging the above relations (introducing the trigonometric functions) the unknown time functions are expressed by

$$S_{1n}(t) = \sum_{i=1}^{2} T_{in}(t) = \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)],$$

$$S_{2n}(t) = \sum_{i=1}^{2} a_{in}T_{in}(t) = \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)]\,a_{in}, \qquad (13)$$

where

$$T_{in}(t) = A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t), \qquad (14)$$

$$a_{in} = (K_1k_n^4 + k - m_1\omega_{in}^2)k^{-1} = k(K_2k_n^4 + k - m_2\omega_{in}^2)^{-1} = \Omega_{10}^{-2}(\Omega_{11n}^2 - \omega_{in}^2)$$

$$= \Omega_{20}^2(\Omega_{22n}^2 - \omega_{in}^2)^{-1}, \quad k_n = l^{-1}n\pi, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots. \qquad (15)$$

It is important to note that the coefficients a_{in} (15) are as follows:

$$a_{1,2n} = 0{\cdot}5\Omega_{10}^{-2}\{(\Omega_{11n}^2 - \Omega_{22n}^2) \pm [(\Omega_{11n}^2 - \Omega_{22n}^2)^2 + 4\Omega_{120}^4]^{1/2}\}, \quad a_{1n} > 0,\, a_{2n} < 0,$$

$$a_{1n}a_{2n} = -m_1m_2^{-1} = -M_1M_2^{-1} = -\Omega_{10}^{-2}\Omega_{20}^2, \quad M_i = m_il = \rho_iF_il.$$

It is proved that the coefficient a_{1n} dependent on lower natural frequency ω_{1n} is always positive while a_{2n} dependent on higher frequency ω_{2n} is always negative.

Finally, the free transverse vibrations of an elastically connected simply supported double-beam complex system are described by the following formulae:

$$w_1(x,t) = \sum_{n=1}^{\infty} X_n(x) S_{1n}(t) = \sum_{n=1}^{\infty} X_n(x) \sum_{i=1}^{2} T_{in}(t) = \sum_{n=1}^{\infty}\sum_{i=1}^{2} X_{1in}(x) T_{in}(t)$$

$$= \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)],$$

$$\tag{16}$$

$$w_2(x,t) = \sum_{n=1}^{\infty} X_n(x) S_{2n}(t) = \sum_{n=1}^{\infty} X_n(x) \sum_{i=1}^{2} a_{in} T_{in}(t) = \sum_{n=1}^{\infty}\sum_{i=1}^{2} a_{in}\, X_n(x) T_{in}(t)$$

$$= \sum_{n=1}^{\infty}\sum_{i=1}^{2} X_{2in}(x) T_{in}(t) = \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)]\, a_{in},$$

where

$$X_{1in}(x) = X_n(x) = \sin(k_n x), \quad X_{2in}(x) = a_{in} X_n(x) = a_{in}\sin(k_n x). \tag{17}$$

The functions $X_{1in}(x)$, $X_{2in}(x)$ are the natural mode shapes of vibration of a beam system corresponding to two sequences of the natural frequencies ω_{in}.

An elastically connected simply supported double-beam complex system executes two kinds of vibrating motions: synchronous vibrations ($a_{1n} > 0$) with lower natural frequencies ω_{1n} and asynchronous vibrations ($a_{2n} < 0$) with higher frequencies ω_{2n}.

The general mode shapes of vibration shown in Figure 2 are the same as the natural mode shapes determined for a doubling-string system [1, 2, 8]. It is also seen that the nature of the free vibrations for a double-beam system is analogous to that for a double-string system. The mathematical form of the corresponding solutions is identical for both systems as a consequence of governing the same boundary conditions.

To find the final form of the free vibrations the initial-value problem is solved. The unknown constants A_{in} and B_{in} are determined from the assumed initial conditions (3) using the orthogonality property of mode shape functions. In this case the classical orthogonality condition is applied:

$$\int_0^l X_m X_n \,\mathrm{d}x = \int_0^l \sin(k_m x)\sin(k_n x)\,\mathrm{d}x = c\delta_{mn}, \tag{18}$$

$$c = c_n^2 = \int_0^l X_n^2\,\mathrm{d}x = \int_0^l \sin^2(k_n x)\mathrm{d}x = 0{\cdot}5l,$$

δ_{mn} is the Kronecker delta function: $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$. Substitution the solutions (16) into the initial conditions (3) gives

$$w_{10} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} B_{in}, \qquad v_{10} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} \omega_{in} A_{in},$$

$$w_{20} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} a_{in} B_{in}, \qquad v_{20} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} a_{in}\omega_{in} A_{in}.$$

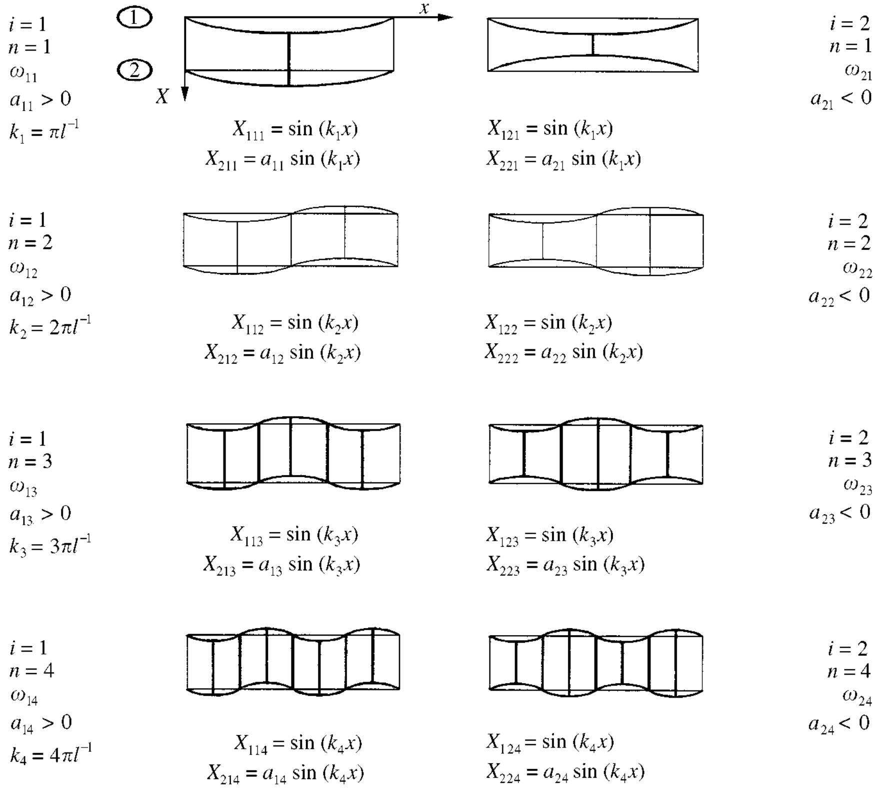

Figure 2. The general mode shapes of vibration of an elastically connected simply supported double-beam complex system corresponding to the first four pairs of the natural frequencies.

Multiplying the above relations by the eigenfunction X_m, then integrating them with respect to x from 0 to l and using the orthogonality condition (18) produces

$$c^{-1}\int_0^l w_{10}X_n\,\mathrm{d}x = \sum_{i=1}^{2} B_{in}, \qquad c^{-1}\int_0^l v_{10}X_n\,\mathrm{d}x = \sum_{i=1}^{2} \omega_{in}A_{in},$$

$$c^{-1}\int_0^l w_{20}X_n\,\mathrm{d}x = \sum_{i=1}^{2} a_{in}B_{in}, \qquad c^{-1}\int_0^l v_{20}X_n\,\mathrm{d}x = \sum_{i=1}^{2} a_{in}\omega_{in}A_{in}.$$

Solving these equations the following formulae making it possible to calculate the unknown constants are obtained:

$$\begin{aligned} A_{1n} &= (c_{1n}\omega_{1n})^{-1}\int_0^l (a_{2n}v_{10} - v_{20})\sin(k_n x)\mathrm{d}x, \\ A_{2n} &= (c_{2n}\omega_{2n})^{-1}\int_0^l (a_{1n}v_{10} - v_{20})\sin(k_n x)\mathrm{d}x, \\ B_{1n} &= c_{1n}^{-1}\int_0^l (a_{2n}w_{10} - w_{20})\sin(k_n x)\,\mathrm{d}x, \\ B_{2n} &= c_{2n}^{-1}\int_0^l (a_{1n}w_{10} - w_{20})\sin(k_n x)\,\mathrm{d}x, \end{aligned} \tag{19}$$

where

$$c_{1n} = -c_{2n} = (a_{2n} - a_{1n})c = 0{\cdot}5 l \Omega_{10}^{-2}(\omega_{1n}^2 - \omega_{2n}^2).$$

4. NUMERICAL EXAMPLE

The free tranverse vibrations of two simply supported beams are considered to establish the effect of physical parameters characterizing the vibrating system on the natural frequencies.

The following values of the parameters are used in the numerical calculations:

$$E = 1 \times 10^{10}\,\mathrm{N\,m^{-2}}, \quad F = 5 \times 10^{-2}\,\mathrm{m^2}, \quad J = 4 \times 10^{-4}\,\mathrm{m^4},$$

$$K = EJ = 4 \times 10^6\,\mathrm{N\,m^2}, \quad k = (0\text{–}5) \times 10^5\,\mathrm{N\,m^{-2}}, \quad l = 10\,\mathrm{m},$$

$$c = 0{\cdot}5;\ 1;\ 2, \quad m = \rho F = 1 \times 10^2\,\mathrm{kg\,m^{-1}}, \quad \rho = 2 \times 10^3\,\mathrm{kg\,m^{-3}}.$$

The problem is solved for three variants of the system which are denoted as

$$\text{V1.} \quad K_1 = K, \quad K_2 = cK, \quad m_1 = m, \quad m_2 = cm,$$

$$\text{V2.} \quad K_1 = K, \quad K_2 = cK, \quad m_i = m, \quad i = 1, 2,$$

$$\text{V3.} \quad K_i = K, \quad i = 1, 2, \quad m_1 = m, \quad m_2 = cm,$$

where c is a positive constant parameter.

The free vibrations of a double-beam system under consideration are described by formulae (16). The mode shape coefficients a_{1n} and natural frequencies ω_{in} are evaluated from the rearranged expressions (12) and (15) for the above three cases of combinations of the basic parameters of the system

V1. $a_{1n} = 1, \quad a_{2n} = -c^{-1}, \quad a_{1n}a_{2n} = -c^{-1},$

$$\omega_{1n}^2 = K k_n^4 m^{-1}, \quad \omega_{2n}^2 = \omega_{1n}^2 + \Omega_0^2, \quad \Omega_0^2 = (1 + c^{-1})\,km^{-1},$$

V2. $a_{in} = (Kk_n^4 + k - m\omega_{in}^2)k^{-1}, \quad a_{1n}a_{2n} = -1.$

$$\omega_{1,2n}^2 = 0{\cdot}5\{[(1 + c)Kk_n^4 + 2k]m^{-1} \mp ([1 - c)Kk_n^4 m^{-1}]^2 + 4k^2m^{-2})^{1/2}\},$$

V3. $a_{i,n} = (Kk_n^4 + k - m\omega_{in}^2)k^{-1}, \quad a_{1n}a_{2n} = -c^{-1},$

$$\omega_{1,2n}^2 = 0{\cdot}5\{(1 + c^{-1})(Kk_n^4 + k)m^{-1} \mp ([(1 - c^{-1})(Kk_n^4 + k)m^{-1}]^2 + 4k^2(cm^2)^{-1})^{1/2}\}.$$

The calculations of a_{in} and ω_{in} are carried out for three values of a parameters c ($c = 0{\cdot}5$; 1; 2) as a function of stiffness modulus k, which is changed in a certain interval $k = (0\text{–}5) \times 10^5\,\mathrm{N\,m^{-2}}$. The results of the calculations are presented in Tables 1–9 and

TABLE 1

Coefficients of natural mode shapes a_{in}

Variant	V1	V1, 2, 3	V1
c	0·5	1	2
a_{1n}		1	
a_{2n}	−2	−1	−0·5

TABLE 2

Coefficients of natural mode shapes a_{in}

Variant		V2				V3			
c		0·5		2		0·5		2	
$k \times 10^{-5}$	n	1	2	1	2	1	2	1	2
1	a_{1n}	1·10	3·39	0·82	0·16	0·58	0·26	1·25	4·44
	a_{2n}	− 0·91	− 0·29	− 1·21	− 6·20	− 3·50	− 7·64	− 0·40	− 0·11
2	a_{1n}	1·05	2·04	0·91	0·30	0·57	0·23	1·13	2·81
	a_{2n}	− 0·95	− 0·49	− 1·10	− 3·36	− 3·52	− 8·79	− 0·44	− 0·18
3	a_{1n}	1·03	1·64	0·94	0·40	0·43	0·20	1·09	2·24
	a_{2n}	− 0·97	− 0·61	− 1·07	− 2·49	− 4·68	− 9·94	− 0·46	− 0·22
4	a_{1n}	1·02	1·46	0·95	0·49	0·35	0·18	1·06	1·94
	a_{2n}	− 0·98	− 0·69	− 1·05	− 2·05	− 5·77	− 10·99	− 0·47	− 0·26
5	a_{1n}	1·02	1·36	0·96	0·55	0·29	0·17	1·05	1·76
	a_{2n}	− 0·98	− 0·74	− 1·04	− 1·80	− 6·86	− 12·05	− 0·48	− 0·28

TABLE 3

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V1; $c = 0{\cdot}5$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{in}	19·7	79·0	177·7	315·8	493·5
1	ω_{2n}	58·2	96·1	185·9	320·5	496·5
2	ω_{2n}	79·9	110·6	193·8	325·2	499·5
3	ω_{2n}	96·9	123·4	201·4	329·8	502·5
4	ω_{2n}	111·3	135·0	208·7	334·3	505·5
5	ω_{2n}	124·1	145·7	215·8	338·7	508·5

in Figures 3–5. On the diagrams the natural frequencies ω_{in} are additionally denoted by subscripts 1, 2, 3, to distinguish the corresponding frequencies computed for $c = 0{\cdot}5$; 1; 2 respectively. If the natural frequency is independent of constant c, then this subindex is not applied.

In general, an elastically connected simply supported double-beam system executes two fundamental kinds of vibrations. The system vibrating with lower natural frequencies

TABLE 4

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V1, 2, 3; $c = 1$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{in}	19·7	79·0	177·7	315·8	493·5
1	ω_{2n}	48·9	90·7	183·2	319·0	495·5
2	ω_{2n}	66·3	101·2	188·6	322·1	497·5
3	ω_{2n}	79·9	110·6	193·8	325·2	499·5
4	ω_{2n}	91·6	119·3	198·9	328·2	501·5
5	ω_{2n}	101·9	127·4	203·9	331·3	503·5

TABLE 5

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V1; $c = 2$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{in}	19·7	79·0	177·7	315·8	493·5
1	ω_{2n}	43·5	87·9	181·8	318·2	495·0
2	ω_{2n}	58·2	96·1	185·9	320·5	496·5
3	ω_{2n}	69·9	103·6	189·9	322·9	498·0
4	ω_{2n}	79·9	110·6	193·8	325·2	499·5
5	ω_{2n}	88·8	117·2	197·6	327·5	501·0

TABLE 6

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V2; $c = 0·5$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{1n}	14·0	56·0	125·9	223·9	349·8
	ω_{2n}	19·1	78·8	177·3	315·3	492·6
1	ω_{1n}	17·0	62·0	129·6	226·0	351·2
	ω_{2n}	47·9	86·6	180·3	316·9	493·6
2	ω_{1n}	17·0	64·4	132·7	228·1	352·6
	ω_{2n}	65·5	95·9	183·6	318·5	494·7
3	ω_{1n}	17·0	65·6	135·3	230·1	353·9
	ω_{2n}	79·3	105·1	187·1	320·3	495·7
4	ω_{1n}	17·1	66·3	137·5	232·0	355·3
	ω_{2n}	91·1	113·8	190·8	322·0	496·8
5	ω_{1n}	17·1	66·7	139·3	233·7	356·6
	ω_{2n}	101·5	122·1	194·7	323·9	497·9

ω_{1n} performs the synchronous vibrations for which the coefficients of mode shapes a_{1n} are always positive ($a_{1n} > 0$). Next, the asynchronous vibrations are executed with higher frequencies ω_{2n} at the mode shape coefficients a_{2n}, which are always negative ($a_{2n} < 0$).

It is seen that in the case $c = 1$ all three variants correspond with the simple system of two physically and geometrically identical beams. For this interesting double-beam

TABLE 7

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V2; $c = 2$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{1n}	19·7	78·9	177·6	315·7	493·2
	ω_{2n}	27·9	111·7	251·3	446·8	698·1
1	ω_{1n}	23·8	84·1	180·3	317·2	494·2
	ω_{2n}	51·0	116·8	253·4	447·9	698·8
2	ω_{1n}	24·0	87·4	182·8	318·7	495·2
	ω_{2n}	67·8	122·7	255·5	449·1	699·6
3	ω_{1n}	24·0	89·6	185·1	320·2	496·2
	ω_{2n}	81·2	129·2	257·8	450·2	700·3
4	ω_{1n}	24·1	91·0	187·2	321·7	497·2
	ω_{2n}	92·7	135·8	260·0	451·4	701·0
5	ω_{1n}	24·1	92·0	189·1	323·1	498·2
	ω_{2n}	102·9	142·3	262·6	452·6	701·8

TABLE 8

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V3; $c = 0{\cdot}5$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{1n}	19·7	79·0	177·7	315·8	493·5
	ω_{2n}	27·9	111·7	251·2	446·6	697·9
1	ω_{1n}	28·6	83·5	180·3	317·4	494·5
	ω_{2n}	60·5	121·4	255·3	448·9	699·3
2	ω_{1n}	35·4	88·2	182·9	318·9	495·5
	ω_{2n}	76·9	130·1	259·3	451·1	700·8
3	ω_{1n}	45·9	92·9	185·4	320·5	496·5
	ω_{2n}	89·8	138·1	263·2	453·4	702·2
4	ω_{1n}	54·8	97·5	188·0	322·0	497·5
	ω_{2n}	100·8	145·6	267·1	455·6	703·6
5	ω_{1n}	62·7	102·0	190·5	323·5	498·5
	ω_{2n}	110·6	152·6	270·9	457·8	705·0

system one has

$$\omega_{1n}^2 = K k_n^4 m^{-1}, \quad \omega_{2n}^2 = \omega_{1n}^2 + \Omega_0^2, \quad \Omega_0^2 = 2k m^{-1}, \quad a_{1n} = 1, \quad a_{2n} = -1.$$

The important conclusions can be drawn from the above expressions. The synchronous natural frequencies ω_{1n} are not dependent on the stiffness modulus k unlike ω_{2n}. The synchronous vibrations are performed by both beams with equal amplitudes ($a_{1n} = 1$), and the lower natural frequencies ω_{1n} are the same as for a single beam. As a consequence of this an elastic layer is not deformed on the transverse direction. The asynchronous vibrations are also performed with equal amplitudes ($a_{2n} = -1$), and the natural frequencies ω_{2n} are

TABLE 9

Natural frequencies of double-beam system $\omega_{in}(s^{-1})$; V3; $c = 2$

$k \times 10^{-5}$	n	1	2	3	4	5
0	ω_{1n}	14·0	55·8	125·6	223·3	348·9
	ω_{2n}	19·8	79·0	177·8	316·1	493·9
1	ω_{1n}	11·8	52·9	123·8	222·2	348·2
	ω_{2n}	44·1	89·8	183·2	319·2	495·9
2	ω_{1n}	11·6	51·5	122·4	221·2	347·5
	ω_{2n}	58·7	98·8	188·2	322·2	497·9
3	ω_{1n}	11·6	50·2	121·1	220·3	346·8
	ω_{2n}	70·4	106·5	192·9	325·2	499·8
4	ω_{1n}	11·5	49·8	120·1	219·4	346·2
	ω_{2n}	80·3	113·5	197·4	328·1	501·8
5	ω_{1n}	11·5	49·3	119·1	218·6	345·6
	ω_{2n}	89·2	120·1	201·7	330·9	503·7

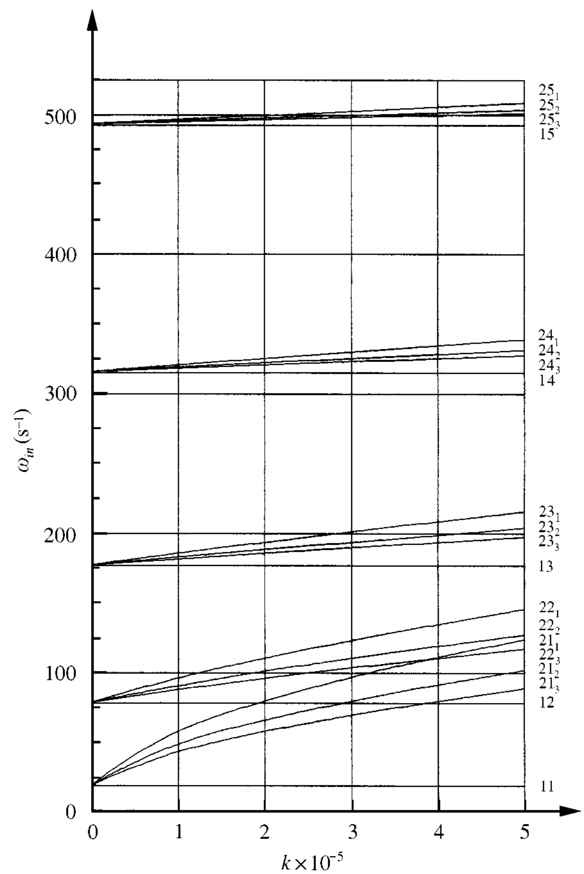

Figure 3. Natural frequencies of double-beam system ω_{in} as a function of elastic layer stiffness modulus k, for various values of a parameter c; variant V1. Notation: in_1, in_2, in_3 denote the graphs of ω_{in} for $c = 0{\cdot}5$; 1; 2 respectively ($i = 1, 2$; $n = 1, 2, \ldots, 5$).

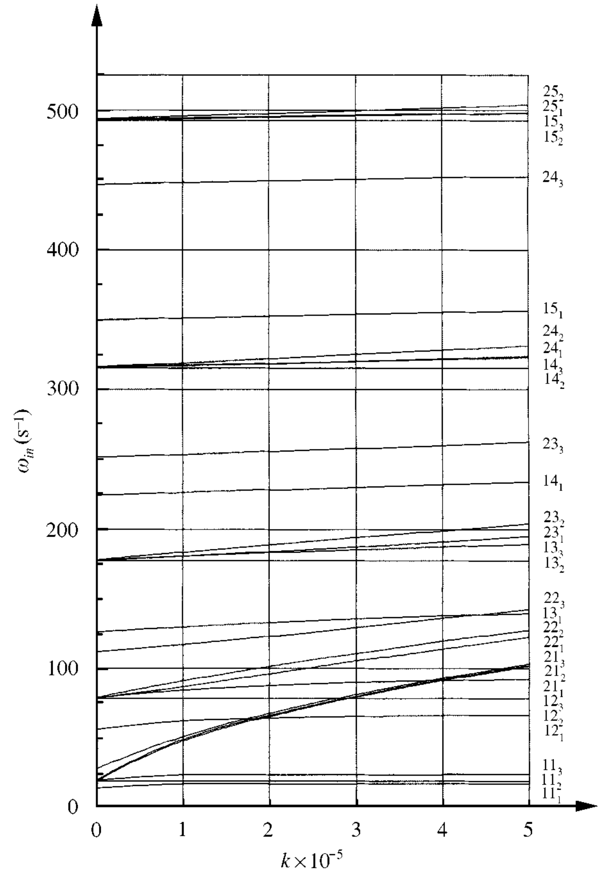

Figure 4. Natural frequencies of double-beam system ω_{in} as a function of elastic layer stiffness modulus k, for various values of a parameter c; variant $V2$. Key as for Figure 3.

identical as for a single beam vibrating on an elastic foundation of stiffness modulus $2k$. The exemplary mode shapes of the free vibrations for this particular case are shown in Figure 6.

From the numerical analysis it is seen that there is a general tendency to increase the natural frequencies ω_{in} in the case of increasing the layer stiffness modulus k for each system variant. The increase for lower natural frequencies is greater than for higher ones. For the variant V1 the simultaneous proportional variation of flexural rigidity and mass of the second beam, implies that the synchronous quantities a_{1n} and ω_{1n} are not dependent on an assumed constant c and layer stiffness modulus k unlike the asynchronous quantities a_{2n} and ω_{2n}. Their values diminish when a parameter c grows. For the variant V2 the beams have the same masses and different flexural rigidities. In general, the change of the second beam flexural rigidity causes the natural frequencies to increase, especially the asynchronous frequencies ω_{2n} for lower harmonics. For the variant V3 the beams have the same flexural rigidities and different masses. The increase of the second beam mass generates the evident reduction of the natural frequencies which is more considerable for the higher harmonics. This effect is greater for the frequencies ω_{2n}.

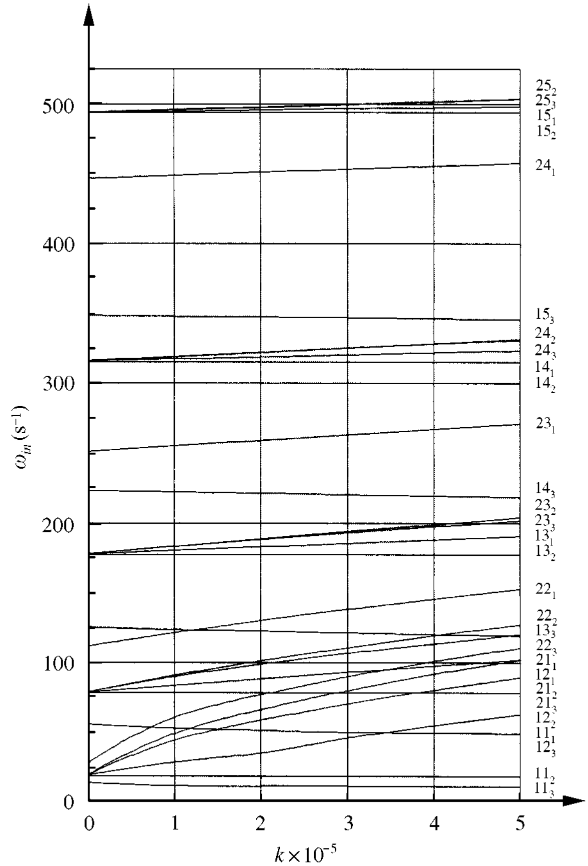

Figure 5. Natural frequencies of double-beam system ω_{in} as a function of elastic layer stiffness modulus k, for various values of a parameter c; variant $V3$. Key as for Figure 3.

5. CONCLUSIONS

The free transverse vibration theory of an elastically connected simply supported double-beam complex system is developed. The solutions of the differential equations of motion are formulated by the classical Bernoulli–Fourier method. The initial-value problem is considered to find the final form of the free vibrations. Two infinite sequences of the natural frequencies ω_{1n}, ω_{2n} ($\omega_{1n} < \omega_{2n}$) are determined. The free vibrations of a double beam are realized by two kinds of motions: synchronous vibrations ($a_{1n} > 0$) with lower natural frequencies ω_{1n} and asynchronous vibrations ($a_{2n} < 0$) with higher frequencies ω_{2n}. The numerical analysis shows the effect of physical parameters of the system on the natural frequencies. It is seen that the nature of the free vibrations for a simply supported double-beam system and for a double-string system [1, 2, 8] is similar. It can be also shown that a corresponding two-degree-of-freedom complex discrete system described in references [1, 2, 9] is an analogue of an elastically connected double-body complex continuous system represented for example by a double-beam system [1, 38, 39, 46, 55, 61], double-string system [1, 2, 8, 9] and double-membrane system [2, 68]. A beam supported on an elastic foundation is a particular case of a double-beam system considered. The solution procedure applied in this paper can be used for the investigation of general elastically connected multi-beam complex system [2, 18, 24, 45].

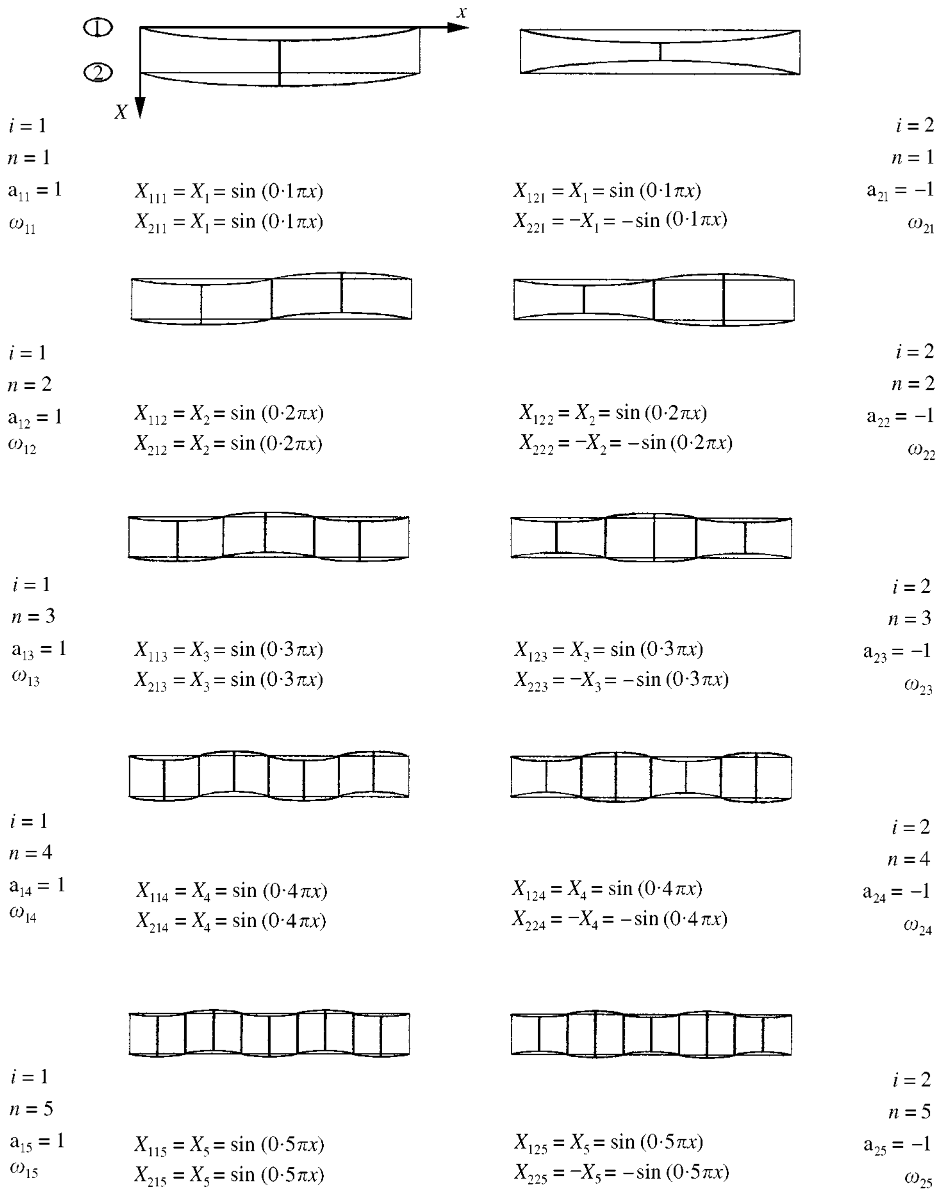

Figure 6. The mode shapes of vibration of an elastically connected simply supported double-beam system corresponding to the first five pairs of the natural frequencies (case $c = 1$).

REFERENCES

1. Z. ONISZCZUK 1996 *Scientific Works of Warsaw University of Technology, Civil Engineering* **130**, 45–65. Transverse vibrations of an elastically connected double-string system (in Polish).
2. Z. ONISZCZUK 1997 *Vibration Analysis of the Compound Continuous Systems with Elastic Constraints*. Rzeszów: Publishing House of Rzeszów University of Technology (in Polish).
3. Z. ONISZCZUK 1998 *Proceedings of the* XVI*th Polish Conference on Theory of Machines and Mechanisms, Rzeszów-Jawor*, Vol. II, 635–642. Transverse vibrations of elastically connected double-string compound system (in Polish).

4. Z. ONISZCZUK 1998 *Proceedings of the* VIII*th Polish Symposium "The Influence of Vibrations on Environment", Kraków-Janowice*, 269–274. The dynamic vibration absorption in the compound continuous system of two solids connected by elastic constraints (in Polish).
5. Z. ONISZCZUK (in press) *Machine Dynamics Problems*. The dynamic vibration absorption in the complex continuous systems.
6. Z. ONISZCZUK 1999 *Proceedings of the* X*th Symposium on Dynamics of Structures, Rzeszów 99, Scientific Works of Rzeszów University of Technology, Mechanics* **174**, 327–332. Free vibration analysis of elastically connected double-string complex system.
7. Z. ONISZCZUK 1999 *Proceedings of the 7th Ukrainian-Polish Seminar "Theoretical Foundations of Civil Engineering", Dnepropetrovsk, Warsaw*, 343–352. Forced vibration analysis of elastically connected double-string complex system.
8. Z. ONISZCZUK 2000 *Journal of Sound and Vibration* **232**, 355–366. Transverse vibrations of elastically connected double-string complex system, Part I: free vibrations.
9. Z. ONISZCZUK 2000 *Journal of Sound and Vibration* **232**, 367–386. Transverse vibrations of elastically connected double-string complex system, Part II: forced vibrations.
10. J. M. SEELIG and W. H. HOPPMANN II 1964 *Journal of the Acoustical Society of America* **36**, 93–99. Normal mode vibrations of systems of elastically connected parallel bars.
11. J. M. SEELIG and W. H. HOPPMANN II 1964 *Transactions of the American Society of Mechanical Engineers, Journal of Applied Mechanics* **31**, 621–626. Impact on an elastically connected double-beam system.
12. P. G. KESSEL 1966 *Journal of the Acoustical Society of America* **40**, 684–687. Resonances excited in an elastically connected double-beam system by a cyclic moving load.
13. P. G. KESSEL and T. F. RASKE 1971 *Journal of the Acoustical Society of America* **49**, 371–373. Damped response of an elastically connected double-beam system due to a cyclic moving load.
14. H. SAITO and S. CHONAN 1968 *Transactions of the Japan Society of Mechanical Engineers* **34**, 1898–1907. Vibrations of elastically connected double-beam systems.
15. H. SAITO and S. CHONAN 1969 *Technology Reports. Tohoku University* **34**, 141–159. Vibrations of elastically connected double-beam systems.
16. A. B. KOZLOV 1968 *Izvestiya Vsesoyusnogo Nautchno-Issledovatelskogo Instituta Ghidrotekhniki* **87**, 192–200. Vibrations of elastically connected bars (in Russian).
17. P. A. KASHIN 1974 *Stroitelnaya Mekhanika, Moscow*, 108–118. Free transverse vibrations of continuously elastically connected beams (in Russian).
18. S. S. RAO 1974 *Journal of the Acoustical Society of America* **55**, 1232–1237. Natural vibrations of systems of elastically connected Timoshenko beams.
19. Y. P. LU and B. E. DOUGLAS 1974 *Journal of Sound and Vibration* **32**, 513–516. On the forced vibrations of a three-layer damped sandwich beams.
20. B. E. DOUGLAS and J. C. S. YANG 1978 *American Institute of Aeronautics and Astronautics Journal* **16**, 925–930. Transverse compressional damping in the vibratory response of elastic–viscoelastic–elastic beams.
21. B. E. DOUGLAS 1986 *Journal of Sound and Vibration* **104**, 343–347. Compressional damping in three-layer beams incorporating nearly incompressible viscoelastic cores.
22. Z. ONISZCZUK 1974 *Journal of Theoretical and Applied Mechanics* **12**, 71–83. Transversal vibration of the system of two beams connected by means of an elastic element (in Polish).
23. Z. ONISZCZUK 1976 *Journal of Theoretical and Applied Mechanics* **14**, 273–282. Free transverse vibrations of an elastically connected double-beam system (in Polish).
24. Z. ONISZCZUK 1977 *Ph.D. Thesis, Cracow University of Technology*. Transverse vibrations of elastically connected double-beam system (in Polish).
25. Z. ONISZCZUK 1979 *Technical Works, Committee of Applied Mechanics, Rzeszów* **III**, 201–227. Tranverse vibrations of elastically connected double-beam system (in Polish).
26. Z. ONISZCZUK 1986 *Proceedings of the* VI*th Symposium on Dynamics of Structures, Rzeszów-Łańcut, Scientific Works of Rzeszów University of Technology, Mechanics* **31**, 161–164. Free transverse vibrations of two beam system connected by nonlinear elastic element (in Polish).
27. Z. ONISZCZUK 1988 *Proceedings of the* XIII*th Symposium "Vibrations in Physical Systems", Poznań-Błażejewko*, 191–192. Free transverse vibrations of the system of two elastically connected multi-span continuous beams.
28. Z. ONISZCZUK 1989 *Proceedings of the PL-YU'89 Polish–Yugoslav Conference on New Trends in Mechanics of Solids and Structures, Rzeszów-Boguchwała*. Forced transverse vibrations of the system of two elastically connected multi-span continuous beams.

29. Z. ONISZCZUK 1989 *Journal of Theoretical and Applied Mechanics* **27**, 347–361. Free transverse vibrations of an elastically connected double-beam system with concentrated masses, elastic and rigid supports (in Polish).
30. Z. ONISZCZUK 1994 *Scientific Works of Rzeszów University of Technology, Mechanics* **126**, 49–71. Forced transverse vibrations of elastically connected double-beam system with concentrated masses, elastic and rigid supports (in Polish).
31. Z. ONISZCZUK 1997 *Proceedings of the 5th Ukrainian-Polish Seminar "Theoretical Foundations of Civil Engineering", Dnepropetrovsk, Warsaw*, 351–360. Free vibrations of elastically connected double-beam system (in Polish).
32. S. CHONAN 1975 *Transactions of the Japan Society of Mechanical Engineers* **41**, 2815–2824. Dynamical behaviours of elastically connected double-beam system subjected to an impulsive load.
33. S. CHONAN 1975 *Bulletin of the Japan Society of Mechanical Engineers* **19**, 595–603. Dynamical behaviours of elastically connected double-beam system subjected to an impulsive load.
34. M. W. HYER, W. J. ANDERSON and R. A. SCOTT 1976 *Journal of Sound and Vibration* **46**, 121–136. Non-linear vibrations of three-layer beams with viscoelastic cores, I: theory.
35. M. W. HYER, W. J. ANDERSON and R. A. SCOTT 1978 *Journal of Sound and Vibration* **61**, 25–30. Non-linear vibrations of three-layer beams with viscoelastic cores, II: experiment.
36. A. V. STEPANOV 1982 *Prikladnaya Mekhanika* **18**, 102–106. Optimization of free vibration damping coefficient in two element continuous system (in Russian).
37. A. S. DMITRIYEV 1983 *Prikladnaya Mekhanika* **19**, 111–115. Dynamics of layered beam structure subjected to a moving concentrated force (in Russian).
38. T. R. HAMADA, H. NAKAYAMA and K. HAYASHI 1983 *Bulletin of the Japan Society of Mechanical Engineers* **26**, 1936–1942. Free and forced vibrations of elastically connected double-beam systems.
39. T. R. HAMADA, H. NAKAYAMA and K. HAYASHI 1983 *Transactions of the Japan Society of Mechanical Engineers* **49**, 289–295. Free and forced vibrations of elastically connected double-beam systems.
40. H. YAMAGUCHI and H. SAITO 1984 *Earthquake Engineering and Structural Dynamics* **12**, 467–479. Vibrations of beams with an absorber consisting of a viscoelastic solid and a beam.
41. A. JOSHI and A. R. UPADHYA 1987 *Journal of Sound and Vibration* **117**, 115–130. Modal coupling effects in the free vibration of elastically interconnected beams.
42. O. SYLWAN 1987 *Journal of Sound and Vibration* **118**, 35–45. Shear and compressional damping effects of constrained layered beams.
43. J. VASWANI, N. T. ASNANI and B. C. NAKRA 1988 *Composite Structures* **10**, 231–245. Vibration and damping analysis of curved sandwich beams with a viscoelastic core.
44. S. S. KOKHMANIUK 1989 *Dinamika Konstruktsiy pri Vozdeystvii Kratkovremennykh Nagruzok*. Kiev: Naukova Dumka (in Russian).
45. Z. YANKELEVSKY 1991 *International Journal of Mechanical Sciences* **33**, 169–177. Analysis of composite layered elastic foundation.
46. T. AIDA, S. TODA, N. OGAWA and Y. IMADA 1992 *Journal of Engineering Mechanics* **118**, 248–258. Vibration control of beams by beam-type dynamic vibration absorbers.
47. Y. FROSTIG and M. BARUCH 1993 *Journal of Engineering Mechanics* **119**, 476–495. High-order buckling analysis of sandwich beams with transversely flexible core.
48. Y. FROSTIG and M. BARUCH 1994 *Journal of Sound and Vibration* **176**, 195–208. Free vibrations of sandwich beams with a transversely flexible core: a high-order approach.
49. S. KUKLA and B. SKALMIERSKI 1994 *Journal of Theoretical and Applied Mechanics* **32**, 581–590. Free vibration of a system composed of two beams separated by an elastic layer.
50. S. KUKLA 1994 *Journal of Sound and Vibration* **172**, 130–135. Free vibration of the system of two beams connected by many translational springs.
51. S. KUKLA 1999 *Proceedings of the Xth Symposium on Dynamics of Structures, Rzeszów 99, Scientific Works of Rzeszów University of Technology, Mechanics* **174**, 171–176. Free vibrations of beam system modelling a telescopic boom (in Polish).
52. M. MACÉ 1994 *Journal of Sound and Vibration* **172**, 577–591. Damping of beam vibrations by means of thin constrained viscoelastic layer: evaluation of a new theory.
53. Y.-H. CHEN and J.-T. SHEU 1994 *Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics* **116**, 350–356. Dynamic characteristics of layered beam with flexible core.

54. Y.-H. CHEN and J.-T. SHEU 1995 *Journal of Engineering Mechanics* **121**, 340–344. Beam on viscoelastic foundation and layered beam.
55. Y.-H. CHEN and C.-Y. LIN 1998 *Journal of Sound and Vibration* **212**, 759–769. Structural analysis and optimal design of a dynamic absorbing beam.
56. G. G. G. LUESCHEN and L. A. BERGMAN 1996 *Journal of Sound and Vibration* **191**, 613–627. Green's function synthesis for sandwiched distributed parameter systems.
57. T. SAKIYAMA, H. MATSUDA and C. MORITA 1996 *Journal of Sound and Vibration* **191**, 189–206. Free vibration analysis of sandwich beam with elastic or viscoelastic core by applying the discrete Green function.
58. T. SAKIYAMA, H. MATSUDA and C. MORITA 1997 *Journal of Sound and Vibration* **203**, 505–522. Free vibration analysis of sandwich arches with elastic or viscoelastic core and various kinds of axis-shape and boundary conditions.
59. W. SZCZEŚNIAK 1996 *Proceedings of the* 4*th Polish-Ukrainian Seminar* "*Theoretical Foundations of Civil Engineering*", *Dnepropetrovsk*, *Warsaw*, Vol. II, 79–105. Vibrations of sandwich beam under moving loads (in Polish).
60. W. SZCZEŚNIAK 1998 *Scientific Works of Warsaw University of Technology*, *Civil Engineering* **132**, 111–151. Vibration of elastic sandwich and elastically connected double-beam system under moving loads (in Polish).
61. K. KAWAZOE, I. KONO, T. AIDA, T. ASO and K. EBISUDA 1998 *Journal of Engineering Mechanics* **124**, 248–258. Beam-type dynamic vibration absorber comprised of free-free beam.
62. K. CABAŃSKA-PŁACZKIEWICZ 1998 *Proceedings of the* XXXVII*th Symposium* "*Modelling in Mechanics*", *Gliwice*, *Scientific Works of Applied Mechanics Department of Silesian University of Technology* **6**, 49–54. Dynamics of system of two Bernoulli-Euler beams with the visco-elastic interlayer (in Polish).
63. K. CABAŃSKA-PŁACZKIEWICZ 1998 *Proceedings of the* XVIII*th Symposium* "*Vibrations in Physical Systems*", *Poznań-Błażejewko*, 83–84. The method of a solution of a problem of vibration of the system of two beams with damping.
64. K. CABAŃSKA-PŁACZKIEWICZ 1998 *Proceedings of the* 6*th Polish-Ukrainian Seminar* "*Theoretical Foundations of Civil Engineering*", *Dnepropetrovsk*, *Warsaw*, 59–68. Free vibrations of system of string-beam with the visco-elastic interlayer (in Polish).
65. K. CABAŃSKA-PŁACZKIEWICZ 1999 *Proceedings of the* 7*th Ukrainian-Polish Seminar* "*Theoretical Foundations of Civil Engineering*", *Dnepropetrovsk*, *Warsaw*, 195–200. The influence of axial forces on free vibration of the system of two viscoelastic beams coupled by a viscoelastic interlayer (in Polish).
66. K. CABAŃSKA-PŁACZKIEWICZ 1999 *Engineering Transactions* **47**, 21–37. Free vibration of the system of two Timoshenko's beams coupled by a viscoelastic interlayer.
67. K. CABAŃSKA-PŁACZKIEWICZ and N. D. PANKRATOVA 1999 *Proceedings of the* XXXVIII*th Symposium* "*Modelling in Mechanics*", *Gliwice*, *Scientific Works of Applied Mechanics. Department of Silesian University of Technology* **9**, 23–28. The dynamic analysis of the system of two beams coupled by an elastic interlayer.
68. Z. ONISZCZUK 1999 *Journal of Sound and Vibration* **221**, 235–250. Transverse vibrations of elastically connected rectangular double-membrane compound system.